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ANALYSIS AND DESIGN OF STRUCTURAL ELEMENTS WITH OPTIMAL LONGEVITY

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Practically all investigations devoted to the optimal design of structural elements are executed under the assumption of steady creep and do not take into account the circumstance that the cumulative damage process accompanied by a continuous redistribution of the stress therein precedes fracture of the material. The solution of optimization problems with the traditional optimality criterion of the equal strength type results in unrealizable designs in the majority of cases.

In this connection, a variational formulation of the problem of analyzing and designing structural elements with optimal longevity is presented below. It is proposed here to use an optimality criterion that takes account of the total damage over the volume of the material during creep as the target functional. A method is developed for solving this problem on the basis of nonlinear programming methods.

Let a body of volume V bounded by a surface S be loaded by surface loads that are constant in time. The system of equations describing creep of the material and simultaneously taking account of the cumulative damage therein has the form [1]

$$\dot{p}_{ij} = \frac{\Phi_1}{(1-\omega)^m}\frac{s_{ij}}{2S_2}, \quad i, j = 1, 2, 3; \tag{1}$$

$$\dot{\omega} = \Phi_2/(1-\omega)^m, \tag{2}$$

where Φ_1, Φ_2 are homogeneous functions in the stress of degree (n + 1) and (g + 1), $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$; σ_{ij} are stress tensor components, $S_2 = s_{ij}s_{ij}/2$, p_{ij} are creep strain tensor components, ω is the damageability parameter, and m, n, g are material characteristics, and the dot denotes differentiation with respect to the time.

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At the initial instant $\omega(x_k, 0) = 0$ at all points of the body, at the time t_* the damageability parameter at a certain point of the body with coordinates x_k^* ($k = 1, 2, 3$) reaches its critical value $\omega(x_k^*, t_*) = 1$ and material fracture occurs (t_* is the time of the beginning of body fracture.

We assume the total strain rate $\dot{\varepsilon}_{ij}$ is the sum of elastic strain $\dot{e}_{ij}$ and creep strain rate tensor components

$$\dot{\varepsilon}_{ij} = \dot{e}_{ij} + \dot{p}_{ij}, \quad i, j = 1, 2, 3. \tag{3}$$

The e_{ij} components are related to σ_{ij} by Hooke's law. For simplicity we consider the material incompressible and in the elastic domain, i.e., $\dot{e}_{ij}\delta_{ij} = 0$.

The equilibrium equations

$$\partial\sigma_{ij}/\partial x_j = 0, \quad i, j = 1, 2, 3 \tag{4}$$

and the Cauchy relationships

$$2\dot{\varepsilon}_{ij} = \partial\dot{u}_i/\partial x_j + \partial\dot{u}_j/\partial x_i, \tag{5}$$

are valid at any point of the body, as are the boundary conditions on the surface S of the body

$$\sigma_{ij}\nu_j = T_i. \tag{6}$$

Here u_i are displacement vector components, ν_j are vector components normal to the body surface S, and T_i are components of the external load vector.

Since the inequality $0 \leq \omega(x_k, t) \leq 1$ is valid for any time, then the total damageability of the material in a volume V at a time t_{**} can be estimated by the functional

$$L = \int_V [1 - \omega(x_k, t_{**})]\, dV, \tag{7}$$

where $\omega(x_k, t_{**})$ is the value of the damageability parameter at the time $t = t_{**}$, the given time of exploitation.

The functional (7) evidently takes on its absolutely minimal value just in the case when the damageability parameter reaches its critical value $\omega(x_k, t_{**}) = 1$ simultaneously at all points of the body. This condition corresponds to an equally strong structure element during creep [2]. It was noted above that realization of equally strong structures does not always turn out to be possible and frequently results in practically unrealizable results. In this connection, we introduce the concept of an optimal structure relative to longevity.

The body (structure element) is called optimal relative to longevity if as much damage as possible is stored in the material up to a previously assigned time t_{**}; in other words, the body is optimal if its material during the exploitation time t_{**} depletes its resources maximally. Therefore, the problem of designing an optimal structure reduces substantially to seeking the minimal value of the functional (7). The stress tensor and strain rate components should here satisfy the relationships (1)-(6).

To take account of the constraints imposed on the structure by the exploitation conditions, the fabrication technology, the allowable dimensions, etc., the system (1)-(7) should be supplemented by the relationships

$$G_l(x_k, \sigma_{ij}, u_i, \omega) \geqslant 0, \; l = 1, 2, 3, \ldots, \tag{8}$$

where G_l are given functions determined by the formulation of the problem.

Therefore, the problem of designing a structure with optimal longevity reduces to investigating the target functional (7) for a conditional extremum with the constraints (1)-(6), (8).

Later we shall consider the following problem: For given external loads and temperature mode determine the geometric dimensions of a structure such that it would be optimal in longevity.

The solution of the nonstationary variational problem (1)-(8) is fraught with significant mathematical difficulties and is possible only by relying on numerical methods and special transformations [3, 4]. This is explained by the fact that the stress-strain state of the structural element depends on the coordinates of the body points and the time, and their determination within the framework of a variational problem is a very complex problem.

In this connection, we reduce the initial variational problem to a nonlinear mathematical programming problem [4, 5]. Using the terminology of the theory of optimal control, we let U denote the vector of the structure parameters to be variated or controlling.

In particular, the controlling parameter is the geometric dimensions of the structure. A state vector of the structure Y corresponds to each value of U. The correspondence between U and Y at any time is set up from the solution of the system (1)-(6).

Superposing a partition mesh of straight lines $x_k^{(n)} = x_k^{(0)} \pm (n-1)\Delta x_k$, e.g., on the volume V, we seek values of U_n at the partition mesh nodes. Replacing the functional (7) by an approximate finite sum and writing the constraints (8) at the partition mesh nodes, by taking account of the notation used we obtain a nonlinear mathematical programming problem instead of the initial variational problem: Minimize the target function

$$L = \sum [1 - \omega(U_n, Y_n)] \Delta x_k^{(n)} \tag{9}$$

while satisfying the constraints

$$G_l(U_n, Y_n) \geqslant 0, \; l = 1, 2, \ldots \tag{10}$$

To solve the optimization problem (9), (10) we use the method of local variations. It is applicable to functionals dependent on several variables, and as is especially important, to complex functionals [4]. The crux of this method is to seek initial allowable values of $U_n^{(0)}$ and alternate changes in these values by the magnitude of the variational step Δh. Only those values $U_n^{(i)} = U_n^{(i-1)} \pm \Delta h$ are taken in each variational step that would result in diminution of the target function (9) and not disturb the constraints (10). Execution of the series of such iterations is continued until the difference between two consecutive values of the target function turns out to be sufficiently small $(|L^{(i)} - L^{(i-1)}| \leqslant \varepsilon)$.

Since the method of local variations generally results in a local extremum, then the result of solving the problem depends a great deal on the number of partition points, the initial allowable values, and the magnitude of the variational step. Consequently, to design structures with optimal longevity it is expedient to use a variable variational step, determined such that those values of U_n having greatest influence on the target function (9) would be subject to change. We will determine the magnitude of the variational step as the controlling parameter diminishes and increases in the i-th iteration at the n-th partition point by the relationships

$$\Delta h_n^{-(i)} = h_0 \left[\frac{\bar{\omega}_{\min}^{(i-1)}(x_k^{(n)}, t_{**})}{\bar{\omega}^{(i-1)}(x_k^{(n)}, t_{**})} \right]^{\lambda}, \; \Delta h_n^{+(i)} = h_0 \left[\frac{\bar{\omega}^{(i-1)}(x_k^{(n)}, t_{**})}{\bar{\omega}_{\max}^{(i-1)}(x_k^{(n)}, t_{**})} \right]^{\lambda}, \tag{11}$$

where h_0 is the greatest value of the variational step, $\bar{\omega}^{(i-1)}$ is the value of the damageability parameter at the point being varied in the (i − 1)-th iteration, $\bar{\omega}_{\min}^{(i-1)}$, $\bar{\omega}_{\max}^{(i-1)}$ are the least and greatest values of the damageability parameter among all the points being varied in the (i − 1)-th iteration, λ is an exponent. Solution of the nonlinear programming problem (1)-(6), (9), (10) is fraught with definite mathematical difficulties and requires colossal expenditures of machine time. This is due to the fact that setting up a connection between the controlling parameters U_n during execution of each iteration and the state parameters Y_n defined by Eqs. (1)-(6) by the traditional method of time steps requires considerable time expenditures in the electronic computer [1, 6]. If it is taken into account that the number of such iterations is large, then the solution of the formulated problem becomes unrealizable, even using modern electronic computers.

Because of the circumstance noted the analysis and design of structures with optimal longevity are expediently performed on the basis of an approximate method elucidated in [7, 8]. According to this method, which is based on utilizing a mixed variational principle, the problem of determining the stress–strain state of a structure reduces to solving an analogous problem under the assumption of steady creep of the material. The desired solution is obtained here for any time by multiplying the solution of the steady creep problem by a function of the coordinates and the time, for whose determination a system of integrodifferential equations is obtained [7, 8] and whose solution can be obtained with minimal time expenditures on the computer. Utilization of the proposed approximate method permits reduction of the original nonstationary optimization problem to a stationary problem, to set up a finite connection at the time t_{**} between the controlling parameters U_n and the state parameters Y_n and thereby substantially cut down the volume of computational operations on the electronic computer.

As an illustration, let us consider the problem of determining the cross-sectional profile of a beam with optimal longevity, of height h and bending moment M for given constraints on the allowable dimensions of the beam width and on the level of the stress state. The time of exploitation t_{**} is considered given.

Taking account of the relationships (7) and (8), the problem formulated is to minimize the functional

$$L = \int_0^{h/2} b [1 - \omega(y, t_{**})] \, dy \tag{12}$$

while satisfying the constraints

$$b_1 \leqslant b(y) \leqslant b_2; \tag{13}$$

$$\sigma(y, t_{**}) < \sigma_T. \tag{14}$$

Here y is the running coordinate along the beam height, b(y) is the width of the beam section, b_1, b_2 are the minimal and maximal allowable dimensions of the beam width, and σ_T the material yield point.

The fundamental relationships of the creep problem (1)-(6) should be satisfied [7]:

$$2\int_0^{h/2} b\sigma y dy = M; \tag{15}$$

$$\dot{\sigma}/E + \dot{p} = \dot{\varkappa} y; \tag{16}$$

$$\dot{p} = B_1\sigma^n/\mu^m; \tag{17}$$

$$\mu = \left[1 - (m+1) B_2 \int_0^t \sigma^{g+1} d\tau\right]^{1/(m+1)}, \tag{18}$$

where for Φ_1, Φ_2 in (1) and (2) there is taken a power-law dependence on the stress $\Phi_1 = B_1\sigma^{n+1}$, $\Phi_2 = B_2\sigma^{g+1}$; $\varkappa$ is the rate of beam curvature, and E is the elastic modulus. The function μ is related to the damageability parameter ω by the relationship

$$\omega = 1 - \mu, \tag{19}$$

obtained by integrating the kinetic damageability equations (2).

We will construct an analysis of the optimal profile on the basis of the mentioned approximate method, which permits reduction of the initial nonstationary optimization problem to a stationary problem, and the method of local variations with a variable variation step.

It is shown in [7] on the basis of a mixed variational principle that the solution of the problem (15)-(19) at any instant has the form

$$\sigma(y, t) = \sigma^0[\mu(y, t)]^{m/n}/X(t). \tag{20}$$

Here σ^0 is the solution of an analogous problem under the assumption of steady creep of the material

$$\sigma^0 = My^{1/n}/2\int_0^{h/2} by^{(n+1)/n} dy. \tag{21}$$

The functions X(t) and u(y, t) are determined here by the expressions

$$X(t) = (1 - t/\bar{t}_*^0)^\gamma; \tag{22}$$

$$\mu^{m/n} = \left\{1 + \frac{\bar{t}_*^0}{t_*^0}\left[\left(1 - \frac{t}{\bar{t}_*^0}\right)^\nu - 1\right]\right\}^\beta, \tag{23}$$

where $t_*^0 = [(m+1) B_2\sigma^{0g+1}]^{-1}$;

$$\bar{t}_*^0 = \left[(m+1) B_2 \int_0^{h/2} b\sigma^{0n+g+2} dy \Big/ \int_0^{h/2} b\sigma^{0n+1} dy\right]^{-1};$$

$$\beta = m/[n + m(n-g-1)];\ \nu = [n + m(n-g-1)]/$$
$$/[n(m+1)];\ \gamma = \beta\nu.$$

Starting from the middle surface and dividing the beam section according to height, into l equal parts and replacing the integrals in (12), (22), (23) by approximate finite sums, then taking account of (19), (20)-(23), we obtain

$$L = \sum_{i=1}^{l+1} b(y_i)\,\mu(y_i, t_{**})\,\Delta y_i; \tag{24}$$

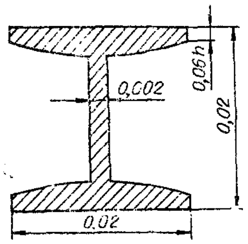

Fig. 1

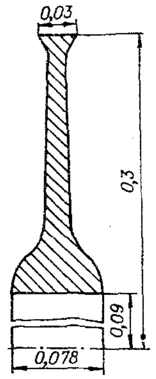

Fig. 2

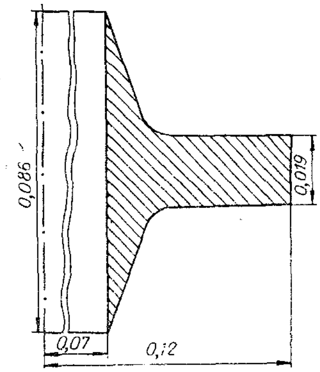

Fig. 3

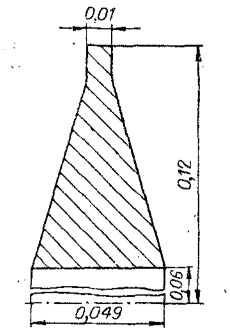

Fig. 4

$$b_1(y_i) \leqslant b(y_i) \leqslant b_2(y_i); \tag{25}$$

$$\sigma^0(y_i)\,\mu^{m/n}(y_i,\ t_{**})/X(t_{**}) < \sigma_T \tag{26}$$

$(\Delta y = h/2l,\ y_i = (i-1)\Delta y)$. Values of $\sigma^0(y_i)$, $\mu(y_i,\ t_{**})$, $X(t_{**})$ at the partition mesh nodes are determined by the relationships (20)-(23).

Being given a certain initial value of the beam profile $b^0(y_i)$ that satisfies the constraints (25) and (26), we determine $L^{(0)}$ and $\mu^{(0)}(y_i,\ t_{**})$ from (23) and (24).

By alternately changing the value of $b^{(0)}(y_i)$ by the magnitude of the variational step $\Delta h^{(1)}(y_i)$ defined by (11), and selecting just those values of $b^{(1)}(y_i)$ that result in diminution of the target function (24) and satisfy the constraints (25) and (26), we obtain $b^{(1)}(y_i)$, $L^{(1)}$, $\mu^{(1)}(y_i,\ t_{**})$. The computation procedure is repeated until the target function (24) reaches its least value; the quantities $b^{(k)}(y_i)$ obtained here will determine the profile of the flexible beam with optimal longevity.

The numerical computation of the optimal profile was performed for a beam of given height h = 0.02 m with fixed shelf thickness of 0.06 h and subjected to a M = 80 Nm bending moment and sustaining given loads for a time of t_{**} = 323 h. The constraints on the beam width and the stress state level governed by (25) and (26), and the material characteristics had the following values [7]: b_1 = 0.002 m, b_2 = 0.02 m, σ_T = 260 MPa, E = 5.6 · 10^4 MPa, n = g = 5, m = 10, B_1 = 1.4043 · 10^{-14} $(\mathrm{MPa})^{-n}$ · h^{-1}, B_2 = 0.9362 · 10^{-15} $(\mathrm{MPa})^{-(g+1)}$ · h^{-1}. The exponent for determining the variational step is λ = 7.

Figure 1 shows the profile of the flexible beam with optimal longevity. It is interesting to note that the geometric dimensions of the beam's rectangular transverse profile that have the same time to fracture under the same loads equal b = 0.01 m, h = 0.02 m. Comparison of these profiles shows thay the weight of the optimal beam is diminished 34% as compared with a beam of rectangular cross section.

Problems of the design of profiles, optimal in longevity, for a rotating disc and flexible and extensible annular plates were considered in an analogous manner.

Figure 2 shows the optimal profile of a disc of radius r_2 = 0.3 m with a hole of radius r_1 = 0.09 m subjected to the action of a p = 40 MPa bucket load, and rotating at a constant angular velocity of n_ω = 5100 rpm.

The exploitation time is $t_{**} = 550$ h. The material characteristics and the constraints on the allowable dimensions of the disc thickness are the following: $E = 6 \cdot 10^4$ MPa, $n = 5.62$, $g = 5$, $m = 9$, $B_1 = 0.3224 \cdot 10^{-12.62}$ $(\text{MPa})^{-n} \cdot \text{h}^{-1}$, $B_2 = 0.3375 \cdot 10^{-13}$ $(\text{MPa})^{-(g+1)} \cdot \text{h}^{-1}$, $\sigma_T = 260$ MPa, $h_1(r_2) = h_2(r_2) = 0.03$ m, $h_1(r) = 0.015$ m, $h_2(r) = 0.08$ m, $\lambda = 4$. It is interesting to note that the time of exploitation of a constant-thickness disc subjected to the action of the very same loads is $t_{**} = 316$ h. Thus it follows that for an identical structure weight the exploitation time of an optimal disc increases 1.74 times as compared with a constant thickness disc.

Figure 3 shows the profile of an optimal plate of radius $r_2 = 0.12$ m with an inner hole of $r_1 = 0.07$ m radius subjected to a bending moment of intensity 8.5 Nm distributed uniformly over the outer contour and sustaining given loads during a time $t_{**} = 276$ h. Here $n = 5$, $g = 5$, $m = 10$, $E = 5.6 \cdot 10^4$ MPa, $B_1 = 0.379 \cdot 10^{-12}$ $(\text{MPa})^{-n} \cdot \text{h}^{-1}$, $B_2 = 0.252 \cdot 10^{-13}$ $(\text{MPa})^{-(g+1)} \cdot \text{h}^{-1}$, $\sigma_T = 260$ MPa, $h_1 = 0.01$ m, $h_2 = 0.1$ m, and $\lambda = 4$.

Comparison of the displayed optimal plate with a constant thickness plate subjected to the action of the very same loads shows that the saving in weight in the optimal design is 15.5% for an identical time to fracture.

Figure 4 shows the profile of an annular plate of optimal longevity, subjected to the action of radial forces $q = -0.145$ MPa · m (uniformly distributed over the inner contour) and sustaining given loads for a time of $t_{**} = 329$ h for $r_2 = 0.12$ m, $r_1 = 0.06$ m, and a material characteristic in conformity with the preceding example. The constraints on the allowable dimensions of the plate thickness were given by the values $h_1 = 0.01$ m and $h_2 = 0.08$ m.

Comparison of the optimal plate represented with a constant thickness plate having the same time to fracture under the same loads shows that the savings in weight is 19.7% in the optimal design.

It follows from the computations presented that the optimal designs correspond completely to real structures and possess substantial advantages here as compared with analogous very simple structures which are expressed either by an increase in the exploitation time or by a diminution in the structure weight.

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